

Locally unitary principal series representations of $\mathrm{GL}_{d+1}(F)$

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To Peter Schneider on the occasion of his 60th birthday

Abstract

For a local field F we consider tamely ramified principal series representations V of $G = \mathrm{GL}_{d+1}(F)$ with coefficients in a finite extension K of \mathbb{Q}_p . Let I_0 be a pro- p -Iwahori subgroup in G , let $\mathcal{H}(G, I_0)$ denote the corresponding pro- p -Iwahori Hecke algebra. If V is locally unitary, i.e. if the $\mathcal{H}(G, I_0)$ -module V^{I_0} admits an integral structure, then such an integral structure can be chosen in a particularly well organized manner, in particular its modular reduction can be made completely explicit.

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1 Introduction

Let F be a local non-Archimedean field with finite residue field k_F of characteristic $p > 0$, let $G = \mathrm{GL}_{d+1}(F)$ for some $d \in \mathbb{N}$. Let K be another local field which is a finite extension of \mathbb{Q}_p , let \mathfrak{o} denote its ring of integers, $\pi \in \mathfrak{o}$ a non-zero element in its maximal ideal and k its residue field.

The general problem of deciding whether a given smooth (or, more generally, locally algebraic) G -representation V over K admits a G -invariant norm — or equivalently: a G -stable free \mathfrak{o} -sub module containing a K -basis of V — is of great importance for the p -adic local Langlands program. It is not difficult to formulate a certain *necessary* condition for the existence of a G -invariant norm on V . This has been emphasized first by Vignéras, see also [2], [3], [6], [7]. If V is a tamely ramified smooth principal series representation

and if $d = 1$ then this condition turns out to also be *sufficient*, see [8]. Unfortunately, if $d > 1$ it is unknown if this condition is sufficient. See however [4] for some recent progress.

In this note we consider tamely ramified smooth principal series representations V of G over K for general $d \in \mathbb{N}$. More precisely, we fix a maximal split torus T , a Borel subgroup P and a pro- p -Iwahori subgroup I_0 in G fixing a chamber in the apartment corresponding to T . We then consider a smooth K -valued character Θ of T which is trivial on $T \cap I_0$, view it as a character of P and form the smooth induction $V = \text{Ind}_P^G \Theta$.

Let $\mathcal{H}(G, I_0)$ denote the pro- p -Iwahori Hecke algebra with coefficients in \mathfrak{o} corresponding to I_0 . The K -subspace V^{I_0} of I_0 -invariants in V is naturally a module over $\mathcal{H}(G, I_0) \otimes_{\mathfrak{o}} K$. The said necessary condition for the existence of a G -invariant norm on V is now equivalent with the condition that the $\mathcal{H}(G, I_0) \otimes_{\mathfrak{o}} K$ -module V^{I_0} admits an integral structure, i.e. an \mathfrak{o} -free $\mathcal{H}(G, I_0)$ -sub module L containing a K -basis of V^{I_0} . One might phrase this as the condition that V be locally integral, or locally unitary.

It is not difficult to directly read off from Θ whether V is locally unitary. (Besides [2] Proposition 3.2 we mention the formulation in terms of Jacquet modules as propagated by Emerton ([3]), see also section 4 below.) We rederive this relationship here. However, the proper purpose of this paper is to provide *explicit* and particularly *well structured* \mathfrak{o} -lattices L_{∇} in V^{I_0} as above whenever V is locally unitary.

Our approach is completely elementary; for example, it does not make use of the integral Bernstein basis for $\mathcal{H}(G, I_0)$ (e.g. [7]). It is merely based on the investigation of certain \mathbb{Z} -valued functions ∇ on the finite Weyl group $W = N(T)/T$, and thus on combinatorics of W . We consider the canonical K -basis $\{f_w\}_{w \in W}$ of V^{I_0} where $f_w \in V^{I_0}$ has support PwI_0 and satisfies $f_w(w) = 1$ (we realize W as a subgroup in G). We then ask for functions $\nabla : W \rightarrow \mathbb{Z}$ such that $L_{\nabla} = \oplus_{w \in W} (\pi)^{\nabla(w)} f_w$ is an \mathfrak{o} -lattice as desired. We show (Theorem 4.2) that whenever V is locally unitary, then V^{I_0} admits an $\mathcal{H}(G, I_0)$ -stable \mathfrak{o} -lattice of this particular shape.

The structure of the $\mathcal{H}(G, I_0)_k = \mathcal{H}(G, I_0) \otimes_{\mathfrak{o}} k$ -modules $L_{\nabla} \otimes_{\mathfrak{o}} k$ so obtained is then encoded in combinatorics of the (finite) Coxeter group W . Approaching them abstractly we suggest the notion of an $\mathcal{H}(G, I_0)_k$ -module of W -type (or: a *reduced standard* $\mathcal{H}(G, I_0)_k$ -module): This is an $\mathcal{H}(G, I_0)_k$ -module $M[\theta, \sigma, \epsilon_{\bullet}]$ with k -basis parametrized by W and whose $\mathcal{H}(G, I_0)_k$ -structure is characterized, by means of some explicit formulae, through a set of data $(\theta, \sigma, \epsilon_{\bullet})$ as follows: θ is a character of $I/I_0 = (T \cap I)/(T \cap I_0)$ where $I \supset I_0$ is the corresponding Iwahori subgroup; σ is a function $\{w \in W \mid \ell(ws_d) > \ell(w)\} \rightarrow \{-1, 0, 1\}$ where s_d is the simple reflection corresponding to an end in the Dynkin diagram, and ℓ is the length function on W ; finally, $\epsilon_{\bullet} = \{\epsilon_w \mid w \in W\}$ is a set of units in k . (But not any such set of data $(\theta, \sigma, \epsilon_{\bullet})$ defines an $\mathcal{H}(G, I_0)_k$ -module $M[\theta, \sigma, \epsilon_{\bullet}]$.)

The explicit nature of $L_{\nabla} \otimes_{\mathfrak{o}} k$, and more generally of an $\mathcal{H}(G, I_0)_k$ -module of W -

type, is particularly well suited for computing its value under a certain functor from finite dimensional $\mathcal{H}(G, I_0)_k$ -modules to (φ, Γ) -modules (if $F = \mathbb{Q}_p$), see [5].

We intend to generalize the results of the present paper to other reductive groups in the future. Moreover, the relationship between $\mathcal{H}(G, I_0)_k$ -modules of W -type (reduced standard $\mathcal{H}(G, I_0)_k$ -modules) and standard $\mathcal{H}(G, I_0)_k$ -modules should be clarified.

The outline is as follows. In section 2 we first introduce the notion of a balanced weight of length $d + 1$: a $(d + 1)$ -tuple of integers satisfying certain boundedness conditions which later on will turn out to precisely encode the condition (on Θ) for V to be locally unitary. Given such a balanced weight, we show the existence of certain functions $\nabla : W \rightarrow \mathbb{Z}$ 'integrating' it. In section 3 we introduce $V = \text{Ind}_P^G \Theta$ and show that if a function ∇ 'integrates' the 'weight' associated with Θ , then L_∇ is an $\mathcal{H}(G, I_0)$ -stable \mathfrak{o} -lattice as desired. In section 4 we put the results of sections 2 and 3 together. In section 5 we introduce $\mathcal{H}(G, I_0)_k$ -modules of W -type.

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2 Functions on symmetric groups

For a finite subset I of $\mathbb{Z}_{\geq 0}$ we put

$$\Delta(I) = \sum_{i \in I} i - \frac{|I| \cdot (|I| - 1)}{2}.$$

Definition: Let $d, r \in \mathbb{N}$. We say that a sequence of integers $(n_i)_{0 \leq i \leq d} = (n_0, \dots, n_d)$ is a balanced weight of length $d + 1$ and amplitude r if $\sum_{i=0}^d n_i = 0$ and if for each subset $I \subset \{0, \dots, d\}$ we have

$$(1) \quad r\Delta(I) \geq \sum_{i \in I} n_i \geq -r\Delta(\{0, \dots, d\} - I).$$

Lemma 2.1. *If $(n_i)_{0 \leq i \leq d}$ is a balanced weight of length $d + 1$ and amplitude r , then so is $(-n_{d-i})_{0 \leq i \leq d}$.*

PROOF: For any $I \subset \{0, \dots, d\}$ we compute

$$\begin{aligned}
\Delta(I) &= \sum_{i \in I} i - \frac{|I| \cdot (|I| - 1)}{2} \\
&= \sum_{i=0}^d i - \sum_{i \notin I} i - d|I| - \frac{|I|^2}{2} + \frac{(d+1)|I| + d|I|}{2} \\
&= \frac{d(d+1)}{2} - \sum_{i \notin I} i - d|I| - \frac{|I|^2}{2} + \frac{(d+1)|I| + d|I|}{2} \\
&= d(d+1 - |I|) - \sum_{i \notin I} i - \frac{(d+1 - |I|)(d - |I|)}{2} \\
&= \sum_{i \notin I} (d - i) - \frac{(d+1 - |I|)(d - |I|)}{2} \\
&= \Delta(\{d - i \mid i \in \{0, \dots, d\} - I\}).
\end{aligned}$$

Together with the assumption $\sum_{i=0}^d n_i = 0$ this shows that the set of inequalities (1) for $(n_i)_{0 \leq i \leq d}$ is equivalent with the same set of inequalities for $(-n_{d-i})_{0 \leq i \leq d}$. Namely, given $I \subset \{0, \dots, d\}$, the inequalities (1) for $(n_i)_{0 \leq i \leq d}$ and I are equivalent with the inequalities (1) for $(-n_{d-i})_{0 \leq i \leq d}$ and $\{d - i \mid i \in \{0, \dots, d\} - I\}$. \square

Lemma 2.2. *Let $(n_i)_{0 \leq i \leq d}$ be a balanced weight of length $d + 1$ and amplitude r .*

(a) *There is a balanced weight $(\tilde{n}_i)_{0 \leq i \leq d}$ of length $d + 1$ and amplitude r such that $\tilde{n}_0 = 0$ and $0 \leq n_i - \tilde{n}_i \leq r$ for all $1 \leq i \leq d$.*

(b) *There is a balanced weight $(m_i)_{0 \leq i \leq d-1}$ of length d and amplitude r such that $0 \leq n_i - m_{i-1} \leq r$ for each $i = 1, \dots, d$.*

PROOF: We first show that (b) follows from (a). Indeed, suppose we are given $(\tilde{n}_i)_{0 \leq i \leq d}$ as in (a). Then put $m_{i-1} = \tilde{n}_i$ for $i = 1, \dots, d$. We clearly have $\sum_{i=0}^{d-1} m_i = 0$. Next, let $I \subset \{0, \dots, d-1\}$. Putting $I^+ = \{i + 1 \mid i \in I\}$ and $I_0^+ = I^+ \cup \{0\}$ we then find

$$\begin{aligned}
r\Delta(I) &= r\left(\sum_{i \in I} i - \frac{|I|(|I| - 1)}{2}\right) \\
&= r\left(\sum_{i \in I_0^+} i - |I| - \frac{|I|(|I| - 1)}{2}\right) \\
&= r\left(\sum_{i \in I_0^+} i - \frac{|I_0^+|(|I_0^+| - 1)}{2}\right) \\
&= r\Delta(I_0^+) \\
&\stackrel{(i)}{\geq} \sum_{i \in I_0^+} \tilde{n}_i = \sum_{i \in I} m_i
\end{aligned}$$

where (i) holds true by assumption. Similarly, we find

$$\begin{aligned}
-r\Delta(\{0, \dots, d-1\} - I) &= -r\left(\sum_{i \in \{0, \dots, d-1\} - I} i - \frac{(d - |I|)(d - |I| - 1)}{2}\right) \\
&= -r\left(\sum_{i \in \{0, \dots, d\} - I^+} i - (d - |I|) - \frac{(d - |I|)(d - |I| - 1)}{2}\right) \\
&= -r\left(\sum_{i \in \{0, \dots, d\} - I^+} i - \frac{(d + 1 - |I^+|)(d - |I^+|)}{2}\right) \\
&= -r\Delta(\{0, \dots, d\} - I^+) \\
&\stackrel{(ii)}{\leq} \sum_{i \in I^+} \tilde{n}_i = \sum_{i \in I} m_i
\end{aligned}$$

where (ii) holds true by assumption.

Now we prove statement (a) in three steps.

Step 1: *For any sequence of integers t_1, \dots, t_d satisfying*

$$(2) \quad r|I|(d - \frac{1}{2}(|I| - 1)) \geq \sum_{i \in I} t_i \geq \frac{1}{2}r|I|(|I| - 1)$$

for each subset $I \subset \{1, \dots, d\}$, there exists another sequence of integers $\tilde{t}_1, \dots, \tilde{t}_d$, again satisfying formula (2) for each $I \subset \{1, \dots, d\}$ and such that $\sum_{i=1}^d \tilde{t}_i = \frac{1}{2}rd(d-1)$ and $0 \leq t_i - \tilde{t}_i \leq r$ for all $1 \leq i \leq d$.

For a subset $I \subset \{1, \dots, d\}$ we write $I^c = \{1, \dots, d\} - I$. Put

$$\delta = \sum_{i=1}^d t_i - \frac{1}{2}rd(d-1).$$

To construct $\tilde{t}_1, \dots, \tilde{t}_d$ as desired, we put $s_i^{(0)} = t_i$ and define inductively sequences $s_1^{(m)}, \dots, s_d^{(m)}$ for $1 \leq m \leq \delta$ such that $0 \leq t_i - s_i^{(m)} \leq r$, such that $0 \leq s_i^{(m-1)} - s_i^{(m)} \leq 1$, such that $\delta - m = \sum_{i=1}^d s_i^{(m)} - \frac{1}{2}d(d-1)$ and such that for any fixed m the sequence $(s_i^{(m)})_i$ satisfies (2) for each subset $I \subset \{1, \dots, d\}$. Once all the $(s_i^{(m)})_i$ are constructed we may put $\tilde{t}_i = s_i^{(\delta)}$.

Suppose $(s_i^{(m)})_i$ have been constructed for some $m < \delta$. Let $I_0 \subset \{1, \dots, d\}$ be maximal such that $\sum_{i \in I_0} s_i^{(m)} = \frac{1}{2}r|I_0|(|I_0| - 1)$. We have

$$(3) \quad s_{i_0}^{(m)} < s_k^{(m)} \quad \text{for each } i_0 \in I_0 \text{ and each } k \in I_0^c.$$

This follows from combining the three formulae

$$\begin{aligned}\sum_{i \in I_0 \cup \{k\}} s_i^{(m)} &\geq \frac{1}{2}r|I_0 \cup \{k\}|(|I_0 \cup \{k\}| - 1) = \frac{1}{2}r|I_0|(|I_0| - 1) + r|I_0|, \\ \sum_{i \in I_0} s_i^{(m)} &= \frac{1}{2}r|I_0|(|I_0| - 1), \\ \sum_{i \in I_0 - \{i_0\}} s_i^{(m)} &\geq \frac{1}{2}r|I_0 - \{i_0\}|(|I_0 - \{i_0\}| - 1) = \frac{1}{2}r|I_0|(|I_0| - 1) - r(|I_0| - 1)\end{aligned}$$

(the first one and the last one holding by hypothesis).

Claim: There is some $k \in I_0^c$ such that $s_k^{(m)} + r > t_k$.

Suppose that, on the contrary, $s_k^{(m)} + r = t_k$ for all $k \in I_0^c$. As $(t_i)_i$ satisfies (2) we then have

$$r|I_0^c|(d - \frac{1}{2}(|I_0^c| - 1)) \geq \sum_{k \in I_0^c} s_k^{(m)} + r$$

or equivalently

$$r|I_0^c|(d - 1 - \frac{1}{2}(|I_0^c| - 1)) \geq \sum_{k \in I_0^c} s_k^{(m)}.$$

On the other hand, as $m < \delta$ we find

$$\begin{aligned}\sum_{k \in I_0^c} s_k^{(m)} &= \left(\sum_{k \in I_0} s_k^{(m)}\right) - \sum_{k \in I_0} s_k^{(m)} \\ &> \frac{1}{2}rd(d - 1) - \frac{1}{2}r|I_0|(|I_0| - 1) \\ &= r \sum_{n=|I_0|}^{d-1} n \\ &= r|I_0^c|(d - 1 - \frac{1}{2}(|I_0^c| - 1)).\end{aligned}$$

Taken together this is a contradiction. The claim is proven.

We choose some $k \in I_0^c$ such that $s_k^{(m)} + r > t_k$ and put $s_k^{(m+1)} = s_k^{(m)} - 1$ and $s_i^{(m+1)} = s_i^{(m)}$ for $i \in \{1, \dots, d\} - \{k\}$.

Claim: $(s_i^{(m+1)})_i$ satisfies the inequality on the right hand side of (2) for each $I \subset \{1, \dots, d\}$.

If $k \notin I$ this follows from the inequality on the right hand side of (2) for I and $(s_i^{(m)})_i$. Similarly, if $\sum_{i \in I} s_i^{(m)} > \frac{1}{2}r|I|(|I| - 1)$ the claim is obvious. Now assume that $k \in I$ and $\sum_{i \in I} s_i^{(m)} = \frac{1}{2}r|I|(|I| - 1)$. We then find some $i_0 \in I_0$ with $i_0 \notin I$, because otherwise $I_0 \subset I$ and hence (since $k \in I$ but $k \notin I_0$) even $I_0 \subsetneq I$, which would contradict the maximality of I_0 as chosen above. Formula (3) gives $s_k^{(m+1)} \geq s_{i_0}^{(m)}$, hence the inequality on the right hand side of (2) for $(I - \{k\}) \cup \{i_0\}$ and $(s_i^{(m)})_i$ implies the inequality on the right hand side of (2) for I and $(s_i^{(m+1)})_i$.

The claim is proven. All the other properties required of $(s_i^{(m+1)})_i$ are obvious from its construction.

Step 2: *The sequence t_1, \dots, t_d defined by $t_i = n_i + r(d - i)$ satisfies formula (2) for each subset $I \subset \{1, \dots, d\}$.*

Indeed, for each $I \subset \{1, \dots, d\}$ the formula (2) for $(t_i)_{1 \leq i \leq d}$ is equivalently converted into the formula (1) for $(n_i)_{1 \leq i \leq d}$ by means of the following equations:

$$\begin{aligned} r|I|(d - \frac{1}{2}(|I| - 1)) &= r\Delta(I) + \sum_{i \in I} r(d - i), \\ \frac{1}{2}r|I|(|I| - 1) &= -r\Delta(\{0, \dots, d\} - I) + \sum_{i \in I} r(d - i). \end{aligned}$$

Step 3: *If for the t_i as in step 2 we choose \tilde{t}_i as in step 1, then the sequence $(\tilde{n}_i)_{0 \leq i \leq d}$ defined by $\tilde{n}_0 = 0$ and $\tilde{n}_i = \tilde{t}_i - r(d - i)$ for $1 \leq i \leq d$ satisfies the requirements of statement (a).*

It is clear that $\tilde{n}_0 = 0$ and $0 \leq n_i - \tilde{n}_i \leq r$ for all $1 \leq i \leq d$, as well as $\sum_{i=0}^d \tilde{n}_i = 0$. It remains to see that $(\tilde{n}_i)_{0 \leq i \leq d}$ satisfies the inequalities (1) for any $I \subset \{0, \dots, d\}$. If $0 \notin I$ then, using the same conversion formulae as in the proof of step 2, this follows from the fact that $(\tilde{t}_i)_{1 \leq i \leq d}$ satisfies formula (1) for each $I \subset \{1, \dots, d\}$. If however $0 \in I$ then we use the property $\sum_{i=0}^d \tilde{n}_i = 0$: it implies that, for $(\tilde{n}_i)_{0 \leq i \leq d}$, the left hand (resp. right hand) side inequality of formula (1) for I is equivalent with the right hand (resp. left hand) side inequality of formula (1) for $\{0, \dots, d\} - I$, thus holds true because the latter holds true — as we just saw. \square

Let W denote the finite Coxeter group of type A_d . Thus, W contains a set $S_0 = \{s_1, \dots, s_d\}$ of Coxeter generators satisfying $\text{ord}(s_i s_{i+1}) = 3$ for $1 \leq i \leq d - 1$ and $\text{ord}(s_i s_{j+1}) = 2$ for $1 \leq i < j \leq d - 1$. Put $\bar{u} = s_d \cdots s_1$. Let $\ell : W \rightarrow \mathbb{Z}_{\geq 0}$ denote the length function.

It is convenient to realize W as the symmetric group of the set $\{0, \dots, d\}$ such that $s_i = (i - 1, i)$ (transposition) for $1 \leq i \leq d$. For $w \in W$ and $1 \leq i \leq d$ we then have

$$(4) \quad \ell(ws_i) > \ell(w) \quad \text{if and only if} \quad w(i - 1) < w(i),$$

see Proposition 1.5.3 in [1].

Let W' denote the subgroup of W generated by s_1, \dots, s_{d-1} . Any element w in W can be uniquely written as $w = \bar{u}^i w'$ for some $w' \in W'$, some $0 \leq i \leq d$. We may thus define $\mu(w) = i$; equivalently, $\mu(w) \in \{0, \dots, d\}$ is defined by asking $\bar{u}^{-\mu(w)} w \in W'$.

Theorem 2.3. *Let $(n_i)_{0 \leq i \leq d}$ be a balanced weight of length $d + 1$ and amplitude r . There exists a function $\nabla : W \rightarrow \mathbb{Z}$ such that for all $w \in W$ we have*

$$(5) \quad \nabla(w) - \nabla(w\bar{u}) = -n_{\mu(w)}$$

and such that for all $s \in S_0$ and $w \in W$ with $\ell(ws) > \ell(w)$ we have

$$(6) \quad \nabla(w) - r \leq \nabla(ws) \leq \nabla(w).$$

PROOF: We argue by induction on d . The case $d = 1$ is trivial. Now assume that $d \geq 2$ and that we know the result for $d - 1$. By Lemma 2.2 we find a balanced weight $(m_i)_{0 \leq i \leq d-1}$ of length d and amplitude r such that $0 \leq n_i - m_{i-1} \leq r$ for each $i = 1, \dots, d$. Put $\bar{u}' = s_{d-1} \cdots s_1$. Define $\mu' : W' \rightarrow \{0, \dots, d-1\}$ by asking that for any $w \in W'$ the element $(\bar{u}')^{-\mu'(w)}w$ of W' belongs to the subgroup generated by s_1, \dots, s_{d-2} . By induction hypothesis there is a function $\nabla' : W' \rightarrow \mathbb{Z}$ with

$$\nabla'(w) - \nabla'(w\bar{u}') = -m_{\mu'(w)}$$

for all $w \in W'$ and

$$\nabla'(w) - r \leq \nabla'(ws) \leq \nabla'(w)$$

for all $w \in W', s \in \{s_1, \dots, s_{d-1}\}$ with $\ell(ws) > \ell(w)$. Writing $w \in W$ uniquely as $w = w'\bar{u}^j$ with $w' \in W'$ and $0 \leq j \leq d$ we define

$$\nabla(w) = \nabla'(w') + \sum_{t=0}^{j-1} n_{\mu(w'\bar{u}^t)}.$$

That this function ∇ satisfies condition (5) for all $w \in W$ is obvious. We now show that it satisfies condition (6) for $s = s_d$ and all $w \in W$ with $\ell(ws_d) > \ell(w)$. Write $w = w'\bar{u}^j$ with $w' \in W'$ and $0 \leq j \leq d$.

If $j = d$ then $w = w'\bar{u}^d = w's_1 \cdots s_d$ so that $\ell(ws_d) < \ell(w)$ (since $w' \in W'$). Thus, for $j = d$ there is nothing to prove.

Now assume $1 \leq j \leq d-1$. We then have $ws_d = w\bar{u}^{-j}s_{d-j}\bar{u}^j = w's_{d-j}\bar{u}^j$ with $w's_{d-j} \in W'$, and we claim that $\ell(ws_d) > \ell(w)$ implies $\ell(w's_{d-j}) > \ell(w')$. Indeed, $\ell(ws_d) > \ell(w)$ means $w(d-1) < w(d)$, by formula (4). As $\bar{u}^j(d) = d-j$ and $(\bar{u}')^j(d-1) = d-1-j$ this implies $w'(d-1-j) < w'(d-j)$, hence $\ell(w's_{d-j}) > \ell(w')$, again by formula (4). The claim is proven.

Moreover, for $0 \leq t \leq j-1$ we have $w's_{d-j}\bar{u}^t = w'\bar{u}^t s_{d-j+t}$ with $s_{d-j+t} \in W'$. This implies $\mu(w's_{d-j}\bar{u}^t) = \mu(w'\bar{u}^t)$. Therefore the claim $\nabla(w) - r \leq \nabla(ws_d) \leq \nabla(w)$ is reduced to the assumption $\nabla'(w') - r \leq \nabla'(w's_{d-j}) \leq \nabla'(w')$.

Finally assume that $j = 0$, i.e. $w = w' \in W'$. Then $\nabla(w) = \nabla'(w)$ and

$$(7) \quad \begin{aligned} \nabla(ws_d) &= \nabla(w\bar{u}'\bar{u}^d) \\ &= \nabla'(w\bar{u}') + \sum_{t=0}^{d-1} n_{\mu(w\bar{u}'\bar{u}^t)}. \end{aligned}$$

Here $\nabla'(w\bar{u}') = \nabla'(w) + m_{\mu'(w)}$ by the assumption on ∇' . On the other hand $\sum_{t=0}^{d-1} n_{\mu(w\bar{u}'\bar{u}^t)} = -n_{\mu(ws_d)}$ as $\sum_{i=0}^d n_i = 0$. Now we claim that $\mu'(w) + 1 = \mu(ws_d)$. Indeed, we have $w(d) = d - \mu(w)$ and hence also $ws_d(d) = d - \mu(ws_d)$ for $w \in W$. Similarly, we have $w(d-1) = d-1 - \mu'(w)$ and hence also $ws_d(d) = w(d-1) = d-1 - \mu'(w)$ for $w \in W'$, and the claim is proven.

Inserting all this transforms the assumption $0 \leq n_{\mu(ws_d)} - m_{\mu(ws_d)-1} \leq r$ into the condition (6) (for $s = s_d$).

We have proven condition (6) for $s = s_d$ and all $w \in W$ with $\ell(ws_d) > \ell(w)$. Condition (6) for all $s \in S_0$ and all $w \in W$ with $\ell(ws) > \ell(w)$ can be checked directly as well. However, alternatively one can argue as follows.

In the setting of section 3 (and in its notations) choose an arbitrary F with residue field \mathbb{F}_q (for an arbitrary q), and choose K/\mathbb{Q}_p and $\pi \in K$ such that our present r satisfies $\pi^r = q$. We use the elements $t_{\bar{u}^i}$ of T (explicitly given by formula (13)) to define the character $\Theta : T \rightarrow K^\times$ by asking that $\Theta(t_{\bar{u}^i}) = \pi^{-n_{i-1}}$ and that $\Theta|_{T \cap I} = \theta$ be the trivial character. (This is well defined as T is the direct product of $T \cap I$ and the free abelian group on the generators $t_{\bar{u}^i}$ for $0 \leq i \leq d$.) The implication (iii) \Rightarrow (ii) in Lemma 3.5, applied to this Θ , shows that what we have proven so far is enough. \square

3 Hecke lattices in principal series representations I

Fix a prime number p . Let K/\mathbb{Q}_p be a finite extension field, \mathfrak{o} its ring of integers and k its residue field.

Let F be a non-Archimedean locally compact field, \mathcal{O}_F its ring of integers, $p_F \in \mathcal{O}_F$ a fixed prime element and $k_F = \mathbb{F}_q$ its residue field with $q = p^{\log_p q} \in p^{\mathbb{N}}$ elements.

Let $G = \mathrm{GL}_{d+1}(F)$ for some $d \in \mathbb{N}$. Let T be a maximal split torus in G , let $N(T)$ be its normalizer. Let P be a Borel subgroup of G containing T , let N be its unipotent radical.

Let X be the Bruhat Tits building of $\mathrm{PGL}_{d+1}(F)$, let $A \subset X$ be the apartment corresponding to T . Let I be an Iwahori subgroup of G fixing a chamber C in A , let I_0 denote its maximal pro- p -subgroup. The (affine) reflections in the codimension-1-faces of C form a set S of Coxeter generators for the affine Weyl group. We view the latter as a subgroup of the extended affine Weyl group $N(T)/T \cap I$. There is an $s_0 \in S$ such that the image of $S_0 = S - \{s_0\}$ in the finite Weyl group $W = N(T)/T$ is the set of simple reflections.

We find elements $u, s_d \in N(T)$ such that $uC = C$ (equivalently, $uI = Iu$, or also

$uI_0 = I_0u$), such that $u^{d+1} \in \{p_F \cdot \text{id}, p_F^{-1} \cdot \text{id}\}$ and such that, setting

$$s_i = u^{d-i} s_d u^{i-d} \quad \text{for } 0 \leq i \leq d$$

the set $\{s_1, \dots, s_d\}$ maps bijectively to S_0 , while $\{s_0, s_1, \dots, s_d\}$ maps bijectively to S ; we henceforth regard these bijections as identifications. Let $\bar{u} = s_d \cdots s_1 \in W \subset G$. Let $\ell : W \rightarrow \mathbb{Z}_{\geq 0}$ be the length function with respect to S_0 .

For convenience one may realize all these data explicitly, e.g. according to the following choice: T consists of the diagonal matrices, P consists of the upper triangular matrices, N consists of the unipotent upper triangular matrices (i.e. the elements of P with all diagonal entries equal to 1). Then W can be identified with the subgroup of permutation matrices in G . Its Coxeter generators s_i for $i = 1, \dots, d$ are the block diagonal matrices

$$s_i = \text{diag}(I_{i-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, I_{d-i})$$

while u is written in block form as

$$u = \begin{pmatrix} & I_d \\ p_F & \end{pmatrix}.$$

(Here I_m , for $m \geq 1$, always denotes the identity matrix in GL_m .) The Iwahori group I consists of the elements of $\text{GL}_{d+1}(\mathcal{O}_F)$ mapping to upper triangular matrices in $\text{GL}_{d+1}(k_F)$, while I_0 consists of the elements of I whose diagonal entries map to $1 \in k_F$.

For $s \in S_0$ let $\iota_s : \text{GL}_2(F) \rightarrow G$ denote the corresponding embedding. For $a \in F^\times$, $b \in F$ put

$$h_s(a) = \iota_s\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right), \quad \nu_s(b) = \iota_s\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right), \quad \delta_s = \iota_s\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right).$$

We realize W as a subgroup of G in such a way that

$$\iota_s\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = s$$

for all $s \in S_0$. Notice that $\text{Im}(\nu_s) \subset N$ for all $s \in S_0$.

Lemma 3.1. (a) For $s \in S_0$ and $a \in F^\times$ we have

$$(8) \quad s\nu_s(a)s = h_s(a^{-1})\nu_s(a)\delta_s s\nu_s(a^{-1}).$$

(b) For $w \in W$ and $s \in S_0$ with $\ell(ws) > \ell(w)$ and for $b \in F$ we have

$$(9) \quad w\nu_s(b)w^{-1} \in N.$$

PROOF: Statement (a) is a straightforward computation inside $\mathrm{GL}_2(F)$. For statement (b) write $s = s_i$ for some $1 \leq i \leq d$. Then the matrix $w\nu_s(b)w^{-1}$ has entry b at the $(w(i-1), w(i))$ -spot (and coincides with the identity matrix at all other spots). As $\ell(ws_i) > \ell(w)$ implies $w(i-1) < w(i)$ by formula (4), this implies $w\nu_s(b)w^{-1} \in N$. \square

Let $\mathrm{ind}_{I_0}^G \mathbf{1}_{\mathfrak{o}}$ denote the \mathfrak{o} -module of \mathfrak{o} -valued compactly supported functions f on G such that $f(ig) = f(g)$ for all $g \in G$, all $i \in I_0$. It is a G -representation by means of the formula $(g'f)(g) = f(gg')$ for $g, g' \in G$. Let

$$\mathcal{H}(G, I_0) = \mathrm{End}_{\mathfrak{o}[G]}(\mathrm{ind}_{I_0}^G \mathbf{1}_{\mathfrak{o}})^{\mathrm{op}}$$

denote the corresponding pro- p -Iwahori Hecke algebra with coefficients in \mathfrak{o} . Then $\mathrm{ind}_{I_0}^G \mathbf{1}_{\mathfrak{o}}$ is naturally a right $\mathcal{H}(G, I_0)$ -module. For a subset H of G we let χ_H denote the characteristic function of H . For $g \in G$ let $T_g \in \mathcal{H}(G, I_0)$ denote the Hecke operator corresponding to the double coset $I_0 g I_0$. It sends $f : G \rightarrow \mathfrak{o}$ to

$$T_g(f) : G \longrightarrow \mathfrak{o}, \quad h \mapsto \sum_{x \in I_0 \backslash G} \chi_{I_0 g I_0}(hx^{-1})f(x).$$

In particular we have

$$(10) \quad T_g(\chi_{I_0}) = \chi_{I_0 g} = g^{-1} \chi_{I_0} \quad \text{if } gI_0 = I_0 g.$$

Let R be an \mathfrak{o} -algebra, let V be a representation of G on an R -module. The submodule of V^{I_0} of I_0 -invariants in V carries a natural (left) action by the R -algebra $\mathcal{H}(G, I_0)_R = \mathcal{H}(G, I_0) \otimes_{\mathfrak{o}} R$, resulting from the natural isomorphism $V^{I_0} \cong \mathrm{Hom}_{R[G]}((\mathrm{ind}_{I_0}^G \mathbf{1}_{\mathfrak{o}}) \otimes_{\mathfrak{o}} R, V)$. Explicitly, for $g \in G$ and $v \in V^{I_0}$ the action of T_g is given as follows: If the collection $\{g_j\}_j$ in G is such that $I_0 g I_0 = \coprod_j I_0 g_j$, then

$$(11) \quad T_g(v) = \sum_j g_j^{-1} v.$$

Let $\overline{T} = (I \cap T)/(I_0 \cap T) = I/I_0$.

Suppose we are given a character $\Theta : T \rightarrow K^\times$ whose restriction $\theta = \Theta|_{I \cap T}$ to $I \cap T$ factors through \overline{T} . As \overline{T} is finite, θ takes values in \mathfrak{o}^\times , hence induces a character (denoted by the same symbol) $\theta : \overline{T} \rightarrow k^\times$. For any $w \in W$ it defines a homomorphism

$$\theta(wh_s(\cdot)w^{-1}) : k_F^\times \rightarrow k^\times, \quad x \mapsto \theta(wh_s(x)w^{-1})$$

and it makes sense to compare it with the constant homomorphism $\mathbf{1}$ taking all elements of k_F^\times to $1 \in k^\times$. Notice in the following that $\theta(wh_s(\cdot)w^{-1}) = \mathbf{1}$ if and only if $\theta(wh_s(\cdot)sw^{-1}) = \mathbf{1}$. For $w \in W$ and $s \in S_0$ put

$$\kappa_{w,s} = \kappa_{w,s}(\theta) = \theta(w\delta_s w^{-1}) \in \{\pm 1\}.$$

Read Θ as a character of P by means of the natural projection $P \rightarrow T$ and consider the smooth principal series representation

$$V = \text{Ind}_P^G \Theta = \{f : G \rightarrow K \text{ locally constant} \mid f(pg) = \Theta(p)f(g) \text{ for } g \in G, p \in P\}$$

with G -action $(gf)(x) = f(xg)$. For $w \in W$ let $f_w \in V$ denote the unique I_0 -invariant function supported on PwI_0 and with $f_w(w) = 1$. It follows from the decomposition $G = \coprod_{w \in W} PwI_0$ that the set $\{f_w\}_{w \in W}$ is a K -basis of the $\mathcal{H}(G, I_0)_K$ -module V^{I_0} .

Lemma 3.2. *Let $w \in W$ and $s \in S_0$, let $a \in \mathcal{O}_F$.*

- (a) *If $\ell(ws) > \ell(w)$ and $a \notin (p_F)$ then $ws\nu_s(a)s \notin PwI_0$.*
- (b) *If $\ell(ws) > \ell(w)$ then $v\nu_s(a)s \notin PwI_0$ for all $v \in W - \{ws\}$.*
- (c) *$v\nu_s(a)s \notin PwI_0$ for all $v \in W - \{w, ws\}$.*

PROOF: We have $\nu_s(\mathcal{O}_F) \subset I_0$. Therefore all statements will follow from standard properties of the decomposition $G = \coprod_{w \in W} PwI_0$, or rather the restriction of this decomposition to $\text{GL}_{d+1}(\mathcal{O}_F)$; notice that this restriction projects to the usual Bruhat decomposition of $\text{GL}_{d+1}(k_F)$.

(a) The assumption $a \notin (p_F)$, i.e. $a \in \mathcal{O}_F^\times$, implies that $ws\nu_s(a)s \in wIsI$, by formula (8). The assumption $\ell(ws) > \ell(w)$ implies $wIsI \subset PwsI = PwsI_0$ by standard properties of the Bruhat decomposition, hence $wIsI \cap PwI_0 = \emptyset$.

(b) Standard properties of the Bruhat decomposition imply $vI_0s \subset PvsI_0 \cup PvI_0$, as well as $vI_0s \subset PvsI_0$ if $\ell(vs) > \ell(v)$. As $\ell(ws) > \ell(w)$ and $v \neq ws$ statement (b) follows.

(c) The same argument as for (b). \square

Lemma 3.3. *Let $w \in W$ and $s \in S_0$. We have*

$$T_s(f_w) = \begin{cases} f_{ws} & : \ell(ws) > \ell(w) \\ qf_{ws} & : \ell(ws) < \ell(w) \text{ and } \theta(wh_s(\cdot)w^{-1}) \neq \mathbf{1} \\ qf_{ws} + \kappa_{ws,s}(q-1)f_w & : \ell(ws) < \ell(w) \text{ and } \theta(wh_s(\cdot)w^{-1}) = \mathbf{1} \end{cases}$$

PROOF: We have $I_0sI_0 = \coprod_a I_0s\nu_s(a)$ where a runs through a set of representatives for k_F in \mathcal{O}_F . For $y \in G$ we therefore compute, using formula (11):

$$\begin{aligned} (T_s(f_w))(y) &= \left(\sum_a \nu_s(a)s f_w \right)(y) \\ &= \sum_a f_w(y\nu_s(a)s). \end{aligned} \tag{12}$$

Suppose first that $\ell(ws) > \ell(w)$. For $a \notin (p_F)$ we then have $ws\nu_s(a)s \notin PwI_0$ by Lemma 3.2, hence $f_w(ws\nu_s(a)s) = 0$. On the other hand $f_w(ws\nu_s(0)s) = f_w(w) = 1$.

Together we obtain $(T_s(f_w))(ws) = 1$. For $v \in W - \{ws\}$ and any $a \in \mathcal{O}_F$ we have $v\nu_s(a)s \notin PwI_0$ by Lemma 3.2, hence $(T_s(f_w))(v) = 0$. It follows that $T_s(f_w) = f_{ws}$.

Now suppose that $\ell(ws) < \ell(w)$. Then $ws\nu_s(a)sw^{-1} \in N$ for any a , by formula (9), hence $f_w(ws\nu_s(a)s) = \theta(ws\nu_s(a)sw^{-1})f_w(w) = 1$. Summing up we get

$$(T_s(f_w))(ws) = \sum_a f_w(ws\nu_s(a)s) = |k_F| = q.$$

To compute $(T_s(f_w))(w)$ we first notice that $f_w(w\nu_s(0)s) = f_w(ws) = 0$. On the other hand, for $a \notin (p_F)$ we find

$$\begin{aligned} f_w(w\nu_s(a)s) &= f_w(wss\nu_s(a)s) \\ &\stackrel{(i)}{=} f_w(wh_s(a^{-1})\nu_s(a)\delta_s s\nu_s(a^{-1})) \\ &= \theta(wh_s(a^{-1})\nu_s(a)\delta_s sw^{-1})f_w(w\nu_s(a^{-1})) \\ &\stackrel{(ii)}{=} \theta(wh_s(a^{-1})\delta_s sw^{-1}) \\ &= \kappa_{ws,s}\theta(wh_s(a^{-1})sw^{-1}). \end{aligned}$$

Here (i) uses formula (8) while (ii) uses $f_w(w\nu_s(a^{-1})) = f_w(w) = 1$ as well as

$$(wh_s(a^{-1})\nu_s(a)\delta_s sw^{-1}) \cdot (wh_s(a^{-1})\delta_s sw^{-1})^{-1} = ws\nu_s(a^{-1})sw^{-1} \in N,$$

formula (9). Now

$$\sum_{a \notin (p_F)} \theta(wh_s(a)sw^{-1}) = \begin{cases} q-1 & : \quad \theta(wh_s(\cdot)w^{-1}) = \mathbf{1} \\ 0 & : \quad \theta(wh_s(\cdot)w^{-1}) \neq \mathbf{1} \end{cases}$$

Thus $\sum_{a \notin (p_F)} f_w(w\nu_s(a)s) = \kappa_{ws,s}(q-1)$ if $\theta(wh_s(\cdot)w^{-1}) = \mathbf{1}$, but $\sum_{a \notin (p_F)} f_w(w\nu_s(a)s) = 0$ if $\theta(wh_s(\cdot)w^{-1}) \neq \mathbf{1}$. We have shown that $(T_s(f_w))(w) = \kappa_{ws,s}(q-1)$ if $\theta(wh_s(\cdot)w^{-1}) = \mathbf{1}$, but $(T_s(f_w))(w) = 0$ if $\theta(wh_s(\cdot)w^{-1}) \neq \mathbf{1}$. Finally, for $v \in W - \{w, ws\}$ and $a \in \mathcal{O}_F$ we have $v\nu_s(a)s \notin PwI_0$ by Lemma 3.2, hence $(T_s(f_w))(v) = 0$. Summing up gives the formulae for $T_s(f_w)$ in the case $\ell(ws) < \ell(w)$. \square

As \bar{u} is the unique element in $W \subset G$ lifting the image of u in $W = N(T)/T$ we have $\bar{u}^{-1}u \in T$. For $w \in W$ we define

$$t_w = w\bar{u}^{-1}uw^{-1} \in T.$$

We record the formulae

$$\bar{u}^{-1}u = t_{\bar{u}^0} = \text{diag}(p_F, I_d),$$

$$(13) \quad t_{\bar{u}^i} = \text{diag}(I_{d-i+1}, p_F, I_{i-1}) \quad \text{for } 1 \leq i \leq d,$$

In particular we notice that $t_w = t_{ws_i}$ for $2 \leq i \leq d$.

Lemma 3.4. *For $w \in W$ we have*

$$(14) \quad T_{u^{-1}}(f_w) = \Theta(t_w)f_{w\bar{u}^{-1}} \quad \text{and} \quad T_u(f_w) = \Theta(t_{w\bar{u}}^{-1})f_{w\bar{u}}.$$

For $w \in W$ and $t \in T \cap I$ we have

$$(15) \quad T_t(f_w) = \theta(wt^{-1}w^{-1})f_w.$$

PROOF: We use formula (10) in both cases: First,

$$(T_{u^{-1}}(f_w))(w\bar{u}^{-1}) = (uf_w)(w\bar{u}^{-1}) = f_w(w\bar{u}^{-1}u) = \Theta(t_w)f_w(w) = \Theta(t_w)$$

but

$$(T_{u^{-1}}(f_w))(v) = (uf_w)(v) = f_w(vu) = \Theta(vu\bar{u}^{-1}v^{-1})f_w(v\bar{u}) = 0$$

for $v \in W - \{w\bar{u}^{-1}\}$, hence the first one of the formulae in (14); the other one is equivalent with it (or alternatively: proven in the same way). Next,

$$(T_t(f_w))(w) = (t^{-1}f_w)(w) = f_w(wt^{-1}) = \theta(wt^{-1}w^{-1})f_w(w) = \theta(wt^{-1}w^{-1}),$$

but

$$(T_t(f_w))(v) = (t^{-1}f_w)(v) = f_w(vt^{-1}) = \theta(vt^{-1}v^{-1})f_w(v) = 0$$

for $v \in W - \{w\}$, hence formula (15). \square

We assume that there is some $r \in \mathbb{N}$ and some $\pi \in \mathfrak{o}$ such that $\pi^r = q$ and such that Θ takes values in the subgroup of K^\times generated by π and \mathfrak{o}^\times . Notice that, given an arbitrary Θ , this can always be achieved after passing to a suitable finite extension of K . Let $\text{ord}_K : K \rightarrow \mathbb{Q}$ denote the order function normalized such that $\text{ord}_K(\pi) = 1$.

Suppose we are given a function $\nabla : W \rightarrow \mathbb{Z}$. For $w \in W$ we put $g_w = \pi^{\nabla(w)}f_w$ and consider the \mathfrak{o} -submodule

$$L_\nabla = L_\nabla(\Theta) = \bigoplus_{w \in W} \mathfrak{o} \cdot g_w$$

of V^{I_0} which is \mathfrak{o} -free with basis $\{g_w \mid w \in W\}$. We ask under which conditions on ∇ it is stable under the action of $\mathcal{H}(G, I_0)$ on V^{I_0} . Consider the formulae

$$(16) \quad \nabla(w) - \nabla(w\bar{u}) = \text{ord}_K(\Theta(t_{w\bar{u}})),$$

$$(17) \quad \nabla(w) - r \leq \nabla(ws) \leq \nabla(w).$$

Lemma 3.5. *The following conditions (i), (ii), (iii) on ∇ are equivalent:*

(i) L_∇ is stable under the action of $\mathcal{H}(G, I_0)$ on V^{I_0} .

(ii) ∇ satisfies formula (16) for any $w \in W$, and it satisfies formula (17) for any $s \in S_0$ and any $w \in W$ with $\ell(ws) > \ell(w)$.

(iii) ∇ satisfies formula (16) for any $w \in W$, and it satisfies formula (17) for $s = s_d$ and any $w \in W$ with $\ell(ws_d) > \ell(w)$.

PROOF: For $t \in T \cap I$ and $w \in W$ it follows from Lemma 3.4 that

$$(18) \quad T_t(g_w) = \theta(wt^{-1}w^{-1})g_w,$$

$$(19) \quad T_{u^{-1}}(g_w) = \pi^{\nabla(w) - \nabla(w\bar{u}^{-1})} \Theta(t_w) g_{w\bar{u}^{-1}},$$

$$(20) \quad T_u(g_w) = \pi^{\nabla(w) - \nabla(w\bar{u})} \Theta(t_{w\bar{u}}^{-1}) g_{w\bar{u}}.$$

For $w \in W$ and $s \in S_0$ it follows from Lemma 3.3 that

$$(21) \quad T_s(g_w) = \begin{cases} \pi^{\nabla(w) - \nabla(ws)} g_{ws} & : \quad \ell(ws) > \ell(w) \\ \pi^{r + \nabla(w) - \nabla(ws)} g_{ws} & : \quad \ell(ws) < \ell(w) \text{ and } \theta(wh_s(\cdot)w^{-1}) \neq \mathbf{1} \\ \pi^{r + \nabla(w) - \nabla(ws)} g_{ws} + \kappa_{ws,s}(\pi^r - 1)g_w & : \quad \ell(ws) < \ell(w) \text{ and } \theta(wh_s(\cdot)w^{-1}) = \mathbf{1} \end{cases}$$

From these formulae we immediately deduce that condition (i) implies both condition (ii) and condition (iii) on ∇ . Now it is known that $\mathcal{H}(G, I_0)$ is generated as an \mathfrak{o} -algebra by the Hecke operators T_t for $t \in T \cap I$ together with $T_{u^{-1}}$, T_u and T_{s_d} . Thus, to show stability of L_∇ under $\mathcal{H}(G, I_0)$ it is enough to show stability of L_∇ under these operators. The above formulae imply that this stability is ensured by condition (iii). Thus (i) is implied by (iii), and a fortiori by (ii). \square

4 Hecke lattices in principal series representations II

In Lemma 3.5 we saw that the (particularly nice) $\mathcal{H}(G, I_0)$ stable \mathfrak{o} -lattices L_∇ in the $\mathcal{H}(G, I_0)_K$ -module V^{I_0} for $V = \text{Ind}_P^G \Theta$ are obtained from functions $\nabla : W \rightarrow \mathbb{Z}$ satisfying the conditions stated there. We now want to explain that the existence of such a function ∇ can be directly read off from Θ . For $0 \leq i \leq d$ put

$$n_i = -\text{ord}_K(\Theta(t_{\bar{u}^{i+1}})).$$

Corollary 4.1. *If $(n_i)_{0 \leq i \leq d}$ is a balanced weight of length $d + 1$ and amplitude r then there exists a function $\nabla : W \rightarrow \mathbb{Z}$ such that L_∇ is stable under the action of $\mathcal{H}(G, I_0)$ on V^{I_0} .*

PROOF: By Theorem 2.3 there exists a function $\nabla : W \rightarrow \mathbb{Z}$ satisfying condition (iii) of Lemma 3.5. Thus we may conclude with that Lemma. \square

Thus we need to decide for which Θ the collection $(n_i)_{0 \leq i \leq d}$ is a balanced weight of length $d+1$ and amplitude r .

We now assume that $F \subset K$. We normalize the absolute value $|\cdot| : K^\times \rightarrow \mathbb{Q}^\times \subset K^\times$ on K (and hence its restriction to F) by requiring $|p_F| = q^{-1}$. Let $\delta : T \rightarrow F^\times$ denote the modulus character associated with P , i.e. $\delta = \prod_{\alpha \in \Phi^+} |\alpha|$ where Φ^+ is the set of positive roots. Let $N_0 = N \cap I$ and

$$T_+ = \{t \in T \mid t^{-1}N_0t \subset N_0\}.$$

The group W acts on the group of characters $\text{Hom}(T, K^\times)$ through its action on T .

Theorem 4.2. *Suppose that for all $w \in W$ and all $t \in T^+$ we have*

$$(22) \quad |((w\Theta)(w\delta^{\frac{-1}{2}})\delta^{\frac{1}{2}})(t)| \leq 1$$

and that the restriction of Θ to the center of G is a unitary character. Then $(n_i)_{0 \leq i \leq d}$ is a balanced weight of length $d+1$ and amplitude r , and L_∇ is stable under the action of $\mathcal{H}(G, I_0)$ on V^{I_0} .

As the center of G is generated by the element $\prod_{j=0}^d t_{\bar{w}^j} = p_F I_{d+1}$ (cf. formula (13)) together with $\mathcal{O}_F^\times \cdot I_{d+1}$, the condition that the restriction of Θ to the center of G be a unitary character is equivalent with the condition

$$(23) \quad \prod_{j=0}^d |\Theta(t_{\bar{w}^j})| = 1.$$

PROOF: (of Theorem 4.2) Recall that, for convenience, we work with the following realization: T is the group of diagonal matrices, P is the group of upper triangular matrices, s_i (for $1 \leq i \leq d$) is the $(i-1, i)$ -transposition matrix and $u = \bar{u} \cdot \text{diag}(p_F, 1, \dots, 1)$. Thus T_+ is the subgroup of T generated by all $t \in \bar{T}$ (viewed as a subgroup of T by means of the Teichmüller character), by the scalar diagonal matrices (the center of G), and by all the matrices of the form $\text{diag}(1, \dots, 1, p_F, \dots, p_F)$. The modulus character is

$$\delta : T \longrightarrow F^\times, \quad \text{diag}(\alpha_0, \dots, \alpha_d) \mapsto \prod_{i=0}^d |\alpha_i|^{d-2i}.$$

Write $\Theta = \text{diag}(\Theta_0, \dots, \Theta_d)$ with characters $\Theta_j : F^\times \rightarrow K^\times$. Reading W as the symmetric group of the set $\{0, \dots, d\}$, formula (22) for $t = \text{diag}(\alpha_0, \dots, \alpha_d)$ reads

$$(24) \quad \left| \prod_{i=0}^d \Theta_{\tau(i)}(\alpha_i) |\alpha_i|^{\tau(i)-i} \right| \leq 1$$

for all permutations τ of $\{0, \dots, d\}$. Asking formula (24) for all $\text{diag}(\alpha_0, \dots, \alpha_d) \in T^+$ is certainly equivalent with asking it for all $\text{diag}(p_F^{-1}, \dots, p_F^{-1}, 1, \dots, 1)$ and for all $\text{diag}(1, \dots, 1, p_F, \dots, p_F)$ (and all τ). This is equivalent with asking

$$(25) \quad |q|^{\Delta(I)} \leq \left| \prod_{j \in I} \Theta_j(p_F) \right| \leq |q|^{-\Delta(\{0, \dots, d\} - I)}$$

for all $I \subset \{0, \dots, d\}$. Indeed, the inequalities on the left hand side of (25) are the inequalities (24) for the $\text{diag}(p_F^{-1}, \dots, p_F^{-1}, 1, \dots, 1)$ and suitable τ . The inequalities on the right hand side of (25) are the inequalities (24) for the $\text{diag}(1, \dots, 1, p_F, \dots, p_F)$ and suitable τ . Now observe that $\Theta_j(p_F) = \Theta(t_{\overline{a}^{d+1-j}})$ and hence $|\Theta_j(p_F)| = |\pi^{\text{ord}(\Theta(t_{\overline{a}^{d+1-j}}))}| = |\pi^{-n_{d-j}}|$ for $0 \leq j \leq d$. We also have $|q| = |\pi^r|$. Together with Lemma 2.2 we recover formula (1). On the other hand, formula (23) is just the property $\sum_{i=0}^d n_i = 0$. We thus conclude with Corollary 4.1. \square

Remarks: (1) We (formally) put $\chi = \Theta\delta^{-\frac{1}{2}}$. Let $\overline{P} \subset G$ denote the Borel subgroup opposite to P . The same arguments as in [3] page 10 show that (at least if χ is regular) for all $w \in W$ the action of T on the Jacquet module $J_{\overline{P}}(V)$ of V (formed with respect to \overline{P}) admits a non-zero eigenspace with character $(w\chi)\delta^{\frac{-1}{2}}$, i.e. with character $(w\Theta)(w\delta^{\frac{-1}{2}})\delta^{\frac{-1}{2}}$. From [3] we then deduce that the conditions in Theorem 4.2 are a necessary criterion for the existence of an integral structure in V .

(2) This necessary criterion has also been obtained in [2]. Moreover, in loc. cit. it is shown (in a much more general context) that it implies the existence of an integral structure in the $\mathcal{H}(G, I_0)$ -module V^{I_0} . The point of Theorem 4.2 is that it explicitly describes a particularly nice such integral structure.

(3) Consider the smooth dual $\text{Hom}_K(V, K)^{\text{sm}}$ of V ; it is isomorphic with $\text{Ind}_P^G \Theta^{-1}\delta$. Our conditions (22) and (23) for Θ are equivalent with the same conditions for $\Theta^{-1}\delta$.

Remark: Suppose we are in the setting of Corollary 4.1 or Theorem 4.2. Let H denote a maximal compact open subgroup of G containing I . Abstractly, H is isomorphic with $\text{GL}_{d+1}(\mathcal{O}_F)$. Let $\mathfrak{o}[H].L_{\nabla}$ denote the $\mathfrak{o}[H]$ -sub module of V generated by L_{∇} , let $(\mathfrak{o}[H].L_{\nabla})^{I_0}$ denote its \mathfrak{o} -sub module of I_0 -invariants. Then one can show (we do not give the proof here) that the inclusion map $L_{\nabla} \rightarrow (\mathfrak{o}[H].L_{\nabla})^{I_0}$ is surjective (and hence bijective). On the one hand this may be helpful for deciding whether V contains an integral structure, i.e. a G -stable free \mathfrak{o} -sub module containing a K -basis of V . On the other hand it implies (in fact: is equivalent with it) that the induced map

$$L_{\nabla} \otimes_{\mathfrak{o}} k \longrightarrow (\mathfrak{o}[H].L_{\nabla}) \otimes_{\mathfrak{o}} k$$

is injective. This might be a useful observation about the $\mathcal{H}(G, I_0)_k$ -module $L_{\nabla} \otimes_{\mathfrak{o}} k$ (which we call an $\mathcal{H}(G, I_0)_k$ -module of W -type in section 5).

5 $\mathcal{H}(G, I_0)_k$ -modules of W -type

We return to the setting of section 3. For $w \in W$ we define

$$\epsilon_w = \epsilon_w(\Theta) = \pi^{-\text{ord}_K(\Theta(t_w))} \Theta(t_w).$$

Let us write $W^{s_d} = \{w \in W \mid \ell(ws_d) > \ell(w)\}$. For a function $\sigma : W^{s_d} \rightarrow \{-1, 0, 1\}$, for $w \in W$ and $i \in \{-1, 0, 1\}$ we understand the condition $\sigma(w) = i$ as a shorthand for the condition $[w \in W^{s_d} \text{ and } \sigma(w) = i]$.

For $w \in W$ we write $\kappa_w = \kappa_{ws_d, s_d}$.

Suppose that the function $\nabla : W \rightarrow \mathbb{Z}$ satisfies the equivalent conditions of Lemma 3.5. Define a function $\sigma : W^{s_d} \rightarrow \{-1, 0, 1\}$ by setting

$$(26) \quad \sigma(w) = \begin{cases} 1 & : & \nabla(ws_d) = \nabla(w) \\ 0 & : & \nabla(w) - r < \nabla(ws_d) < \nabla(w) \\ -1 & : & \nabla(w) - r = \nabla(ws_d) \end{cases}$$

The action of $\mathcal{H}(G, I_0)$ on L_∇ induces an action of $\mathcal{H}(G, I_0)_k = \mathcal{H}(G, I_0) \otimes_{\mathfrak{o}} k$ on $L_\nabla \otimes_{\mathfrak{o}} k$. The \mathfrak{o} -basis $\{g_w \mid w \in W\}$ of L_∇ induces a k -basis $\{g_w \mid w \in W\}$ of $L_\nabla \otimes_{\mathfrak{o}} k = L_\nabla(\Theta) \otimes_{\mathfrak{o}} k$ (we use the same symbols g_w).

Corollary 5.1. *The action of $\mathcal{H}(G, I_0)_k$ on $L_\nabla \otimes_{\mathfrak{o}} k$ is characterized through the following formulae: For $t \in T \cap I$ and $w \in W$ we have*

$$(27) \quad T_t(g_w) = \theta(wt^{-1}w^{-1})g_w,$$

$$(28) \quad T_{u^{-1}}(g_w) = \epsilon_w g_{w\bar{u}^{-1}} \quad \text{and} \quad T_u(g_w) = \epsilon_{w\bar{u}}^{-1} g_{w\bar{u}},$$

$$(29) \quad T_{s_d}(g_w) = \begin{cases} g_{ws_d} & : & [\sigma(ws_d) = -1 \text{ and } \theta(wh_{s_d}(\cdot)w^{-1}) \neq \mathbf{1}] \text{ or } \sigma(w) = 1 \\ -\kappa_w g_w & : & \sigma(ws_d) \in \{0, 1\} \text{ and } \theta(wh_{s_d}(\cdot)w^{-1}) = \mathbf{1} \\ g_{ws_d} - \kappa_w g_w & : & \sigma(ws_d) = -1 \text{ and } \theta(wh_{s_d}(\cdot)w^{-1}) = \mathbf{1} \\ 0 & : & \text{all other cases} \end{cases}$$

PROOF: Formula (27) follows from formula (18). The assumption $\nabla(w\bar{u}^{-1}) - \nabla(w) = \text{ord}_K(\theta(t_w))$ implies that the formulae in (28) follow from formulae (19) and (20). Finally, formula (29) follows from formula (21) by a case by case checking. \square

Forgetting their origin from some Θ and ∇ , we formalize the structure of $\mathcal{H}(G, I_0)_k$ -modules met in Corollary 5.1 in an independent definition.

Definition: We say that an $\mathcal{H}(G, I_0)_k$ -module M is of W -type (or: a *reduced standard module*) if it is of the following form $M = M(\theta, \sigma, \epsilon_\bullet)$. First, a k -vector space basis of M is the set of formal symbols g_w for $w \in W$. The $\mathcal{H}(G, I_0)_k$ -action on M is characterized by a character $\theta : \overline{T} \rightarrow k^\times$ (which we also read as a character of $T \cap I$ by inflation), a map $\sigma : W^{s_d} \rightarrow \{-1, 0, 1\}$ and a set $\epsilon_\bullet = \{\epsilon_w\}_{w \in W}$ of units $\epsilon_w \in k^\times$. Namely, for $w \in W$ we define $\kappa_w = \kappa_w(\theta) = \theta(ws_d \delta_{s_d} s_d w^{-1}) \in \{\pm 1\}$. Then it is required that for $t \in T \cap I$ and $w \in W$ formulae (27), (28) and (29) hold true.

Conversely we may begin with a character $\theta : \overline{T} \rightarrow k^\times$, a map $\sigma : W^{s_d} \rightarrow \{-1, 0, 1\}$ and a set $\epsilon_\bullet = \{\epsilon_w\}_{w \in W}$ of units $\epsilon_w \in k^\times$ and ask:

Question 1: For which set of data $\theta, \sigma, \epsilon_\bullet$ do formulae (27), (28) and (29) define an action of $\mathcal{H}(G, I_0)_k$ on $\oplus_{w \in W} k.g_w$?

Question 2: For which set of data $\theta, \sigma, \epsilon_\bullet$ does there exist some $\mathcal{H}(G, I_0)$ -module $L_\nabla(\Theta)$ as in Corollary 5.1 such that $L_\nabla(\Theta) \otimes_\bullet k \cong M(\theta, \sigma, \epsilon_\bullet)$ as an $\mathcal{H}(G, I_0)_k$ -module?

In question 2 we regard θ as taking values in $\mathfrak{o}^\times \subset K^\times$ by means of the Teichmüller lifting. Clearly those $\theta, \sigma, \epsilon_\bullet$ asked for in question 2 belong to those $\theta, \sigma, \epsilon_\bullet$ asked for in question 1.

We do not consider question 1 in general, but provide a criterion for a positive answer to question 2. Suppose we are given a set of data $\theta, \sigma, \epsilon_\bullet$ as above.

Proposition 5.2. *Suppose that $\epsilon_w = \epsilon_{ws_i}$ for all $2 \leq i \leq d$ and that there exists a function $\partial : W \rightarrow [-r, r] \cap \mathbb{Z}$ with the following properties:*

$$\sigma(w) = \begin{cases} 1 & : & w \in W^{s_d} \text{ and } \partial(w) = 0 \\ 0 & : & w \in W^{s_d} \text{ and } 0 < \partial(w) < r \\ -1 & : & w \in W^{s_d} \text{ and } \partial(w) = r \end{cases}$$

$$\partial(ws_d) = -\partial(w)$$

$$(30) \quad \partial(w\bar{u}^{d-i}) + \partial(ws_i\bar{u}^{d-j}) = \partial(w\bar{u}^{d-j}) + \partial(ws_j\bar{u}^{d-i})$$

for $1 \leq i < j - 1 < d$, and

$$(31) \quad \partial(w\bar{u}^{d-i}) + \partial(ws_i\bar{u}^{d-i-1}) + \partial(ws_i s_{i+1}\bar{u}^{d-i}) = \partial(w\bar{u}^{d-i-1}) + \partial(ws_{i+1}\bar{u}^{d-i}) + \partial(ws_{i+1} s_i \bar{u}^{d-i-1})$$

for $1 \leq i < d$.

Then there exists an extension $\Theta : T \rightarrow K^\times$ of θ and a function $\nabla : W \rightarrow \mathbb{Z}$ as before such that we have an isomorphism of $\mathcal{H}(G, I_0)_k$ -modules $L_\nabla(\Theta) \otimes_\bullet k \cong M(\theta, \sigma, \epsilon_\bullet)$.

PROOF: *Step 1:* Let $w, v \in W$. Choose a (not necessarily reduced) expression $v = s_{i_1} \cdots s_{i_r}$ (with $i_m \in \{1, \dots, d\}$) and put

$$\partial(w, v) = \sum_{m=1}^r \partial(ws_{i_1} \cdots s_{i_{m-1}} \bar{u}^{d-i_m}).$$

Claim: This definition does not depend on the chosen expression $s_{i_1} \cdots s_{i_r}$ for v .

Indeed, it follows from hypothesis (30) that for $1 \leq i < j-1 < d$ we have $\partial(w, s_i s_j) = \partial(w, s_j s_i)$ where on either side we use the expression of $s_i s_j = s_j s_i$ as indicated. Similarly, it follows from hypothesis (31) that for $1 \leq i < d$ we have $\partial(w, s_i s_{i+1} s_i) = \partial(w, s_{i+1} s_i s_{i+1})$ where on either side we use the expression of $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ as indicated. Finally, for $1 \leq i \leq d$ we have $\partial(w, s_i s_i) = 0$ where we use the expression $s_i s_i$ for the element $s_i s_i = s_i^2 = 1 \in W$: this follows from the definition of ∂ and from $s_i \bar{u}^{d-i} = \bar{u}^{d-i} s_d$. Thus we see that our definition of $\partial(w, v)$ (viewed as a function in $v \in W$, with fixed $w \in W$) respects the defining relations for the Coxeter group W . Iterated application implies the stated claim.

Step 2: The definition of $\partial(w, v)$ implies $\partial(w, v) + \partial(wv, x) = \partial(w, vx)$ for $v, w, x \in W$. Therefore there is a function $\nabla : W \rightarrow \mathbb{Z}$, uniquely determined up to addition of a constant function $W \rightarrow \mathbb{Z}$, such that

$$\nabla(w) - \nabla(wv) = \partial(w, v) \quad \text{for all } v, w \in W.$$

It has the following properties. First, it fulfils formula (26). Next, we have

$$(32) \quad \nabla(w) - \nabla(w\bar{u}) = \nabla(ws_i) - \nabla(ws_i\bar{u}) \quad \text{for } w \in W \text{ and } 1 \leq i \leq d-1.$$

$$(33) \quad \nabla(w\bar{u}^{-1}) - \nabla(w) = \nabla(w\bar{u}^{-1}s_i) - \nabla(ws_i) \quad \text{for } w \in W \text{ and } 2 \leq i \leq d.$$

These formulae are equivalent, as $s_i \bar{u} = \bar{u} s_{i+1}$ for $1 \leq i \leq d-1$. To see that they hold true we compute

$$\begin{aligned} \nabla(w) - \nabla(ws_i) &= \partial(w, s_i) \\ &= \partial(w\bar{u}^{d-i}) \\ &= \partial(w\bar{u}, s_{i+1}) \\ &= \nabla(w\bar{u}) - \nabla(w\bar{u}s_{i+1}) \\ (34) \quad &= \nabla(w\bar{u}) - \nabla(ws_i\bar{u}) \end{aligned}$$

and formula (32) follows.

Step 3: For $w \in W$ we define

$$\Theta(t_w) = \pi^{\nabla(w\bar{u}^{-1}) - \nabla(w)} \epsilon_w \in K^\times.$$

Formula (33) together with our assumption on the ϵ_w implies that this is well defined, because for $w, w' \in W$ we have $t_w = t_{w'}$ if and only if $w^{-1}w'$ belongs to the subgroup of W generated by s_2, \dots, s_d . As $T/T \cap I$ is freely generated by the t_w this defines a character $\Theta : T \rightarrow K^\times$ extending $T \cap I \rightarrow \overline{T} \xrightarrow{\theta} k^\times \subset K^\times$, as desired. \square

Corollary 5.3. *Assume that $d \leq 2$. If we have $\epsilon_w = \epsilon_{ws_i}$ for all $2 \leq i \leq d$ then there exists an extension $\Theta : T \rightarrow K^\times$ of θ and a function $\nabla : W \rightarrow \mathbb{Z}$ such that we have an isomorphism of $\mathcal{H}(G, I_0)_k$ -modules $L_\nabla(\Theta) \otimes_{\mathfrak{o}} k \cong M(\theta, \sigma, \epsilon_\bullet)$.*

PROOF: Choose a function $\partial : W^{s_d} \rightarrow [0, r] \cap \mathbb{Z}$ such that

$$\partial(w) = 0 \text{ if } \sigma(w) = 1, \quad 0 < \partial(w) < r \text{ if } \sigma(w) = 0, \quad \partial(w) = r \text{ if } \sigma(w) = -1.$$

Extend ∂ to a function $\partial : W \rightarrow [-r, r] \cap \mathbb{Z}$ by setting $\partial(ws_d) = -\partial(w)$ for $w \in W^{s_d}$. Then, as we assume $d \leq 2$, properties (30) and (31) are empty resp. fulfilled for trivial reasons. Therefore we conclude with Proposition 5.2. \square

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